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# DIHEDRAL COVERINGS OF ALGEBRAIC SURFACES AND THEIR APPLICATION

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Abstract. In this article, we study dihedral coverings of algebraic surfaces branched along curves with at most simple singularities. A criterion for a reduced curve to be the branch locus of some dihedral covering is given. As an application we have the following:

Let $B$ be a reduced plane curve of even degree $d$ having only $a$ nodes and $b$ cusps. If $2a + 6b > 2d^2 - 6d + 6$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.

Note that Nori's result implies that $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian, provided that $2a + 6b < d^2$.

## Introduction

Let $\Sigma$ be a smooth surface, and $S$ a normal surface equipped with a finite morphism $\pi : S \to \Sigma$. We call $S$ a $\mathcal{D}_{2n}$ covering if the rational function field, $\mathbf{C}(S)$, of $S$ is a Galois extension of $\mathbf{C}(\Sigma)$ with Galois group $\mathcal{D}_{2n}$: the dihedral group of order $2n$. In previous articles [T1], [T2] and [T4], we developed a general theory for $\mathcal{D}_{2n}$ coverings and studied such coverings of $\mathbf{P}^2$. In this article, we continue to study the problem of existence of a $\mathcal{D}_{2n}$ covering possessing a given branch locus. More precisely, we consider the following:

**Question 0.1.** (*a*) *Let $B$ be a reduced divisor on $\Sigma$ with at most simple singularities. Give a condition for $B$ to be the branch locus of some $\mathcal{D}_{2n}$ covering $\pi : S \to \Sigma$ which is branched along $B$. Here simple singularities of curves are those in* II, §8, [BPV].

(*b*) (*A weak version of* (*a*).) *Let $B$ be as above. Give a condition for $B$ to be the branch locus of some $\mathcal{D}_{2n}$ covering $\pi : S \to \Sigma$ which is branched along $B$ with ramification index* 2. *Here, the ramification index means one along the smooth part of $B$.*

Note that, if $\Sigma$ is simply connected and $B$ is irreducible, then any $\mathcal{D}_{2n}$ covering is branched along $B$ with ramification index 2.

In [T1] and [T2], we studied Question 0.1 (a) in the case where $\Sigma = \mathbf{P}^2$ and $degB = 4, 5$; and gave a characterization of possible branch curves in terms of the combination of singularities. In [T4], we considered the same question in the case where $\Sigma = \mathbf{P}^2$ and $B$ is an irreducible *maximizing* sextic curve with a triple point. In this article, we consider Question 0.1 (b) under more general settings as follows:

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As we will see in §1, if the $\mathcal{D}_{2n}$ covering in Question 0.1 (b) exists, then there exists a finite double covering $f' : Z' \to \Sigma$ branched along $B$. This shows that as far as we consider Question 0.1(b), there is no harm to assume the existence of $Z'$. We denote its canonical resolution by $Z$. By the definition of the canonical resolution (see [H]), $Z$ satisfies the following diagram:

$$\begin{array}{ccc} Z' & \xleftarrow{\ g\ } & Z \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ \Sigma & \xleftarrow{\ q\ } & \hat{\Sigma}, \end{array}$$

where $q : \hat{\Sigma} \to \Sigma$ is a composition of blowing-ups so that the induced morphism $f$ is finite. We also denote the composite morphism $q \circ f$ by $\tilde{f}$. Note that $Z'$ is determined by $B$ and a line bundle $\mathcal{L}$ on $\Sigma$ such that $2\mathcal{L} \sim B$ (see [BPV], I.17).

In this article, we study Question 0.1 (b) under the assumptions:

**Assumption 0.2.** The Néron-Severi group, $\mathrm{NS}(Z)$, of $Z$ is torsion free.

Let $T$ be a subgroup of $\mathrm{NS}(Z)$ generated by $\tilde{f}^* \mathrm{NS}(\Sigma)$ and all the irreducible components of the exceptional divisor of the resolution $g : Z \to Z'$. Now we are in a position to state our results.

**Theorem 0.3.** *Let $B$ be a curve as in Question* 0.1(*b*) *and suppose that $Z$ satisfies Assumption* 0.2. *Let $n$ be a fixed positive odd integer. If $\mathrm{NS}(Z)/T$ has a torsion of order $n$ and $\gcd(n, \mathrm{disc}\,(\tilde{f}^* \mathrm{NS}(\Sigma))) = 1$, where $\mathrm{disc}\,(\tilde{f}^* \mathrm{NS}(\Sigma))$ is the determinant of the intersection matrix of the lattice $\tilde{f}^* \mathrm{NS}(\Sigma)$, then there exists a $\mathcal{D}_{2n}$ covering of $\Sigma$ branched along $B$ with ramification index* 2.

Using Theorem 0.3, we give a criterion for the existence of a $\mathcal{D}_{2p}$ ($p$: odd prime) covering in terms of the number of singularities. To state our result, we need to define the total Milnor number (or the index) of $B$. For a singularity $x \in \mathrm{Sing}(B)$, we denote its Milnor number by $\mu_x$. For simple singularities, $\mu_x =$ the subindex of the type of $x$, where we use the notation $a_n$, $d_n$ and $e_n$ to describe the types of simple singularities. The total Milnor number, $\mu(B)$, of $B$ is $\mu(B) = \sum_{x \in \mathrm{Sing}(B)} \mu_x$.

**Theorem 0.4.** *Let $B$ be a curve as in Question* 0.1(*b*) *and suppose that $Z$ satisfies Assumption* 0.2. *Let $\mathcal{L}$ be the line bundle as above. Let $p$ be a fixed odd prime. Define a non-negative integer $l$ to be*

*$l =$ the number of singularities of type $a_{3k-1}$ $(k > 0)$ and $e_6$, if $p = 3$, and;*

*$l =$ the number of singularities of type $a_{pk-1}$ $(k > 0)$, if $p \geq 5$.*

*If $p \not| \mathrm{disc}\,(\tilde{f}^* \mathrm{NS}(\Sigma))$ and*

$$\begin{aligned} l > {} & 24\chi(\mathcal{O}_\Sigma) + 4\left(h^1(\mathcal{O}_\Sigma) + h^1(\mathcal{O}_\Sigma(-\mathcal{L}))\right) \\ & + 4\mathcal{L}^2 + 2K_\Sigma\mathcal{L} - 2K_\Sigma^2 - 2 - \rho(\Sigma) - \mu(B), \end{aligned}$$

*where $K_\Sigma$ is the canonical bundle of $\Sigma$, then there exists a $\mathcal{D}_{2p}$ covering of $\Sigma$ branched along $B$ with ramification index* 2.

Note that Theorems 0.3 and 0.4 are generalization of the corresponding results (Theorem 0.6 and Proposition 3.1) in [T4].

We apply Theorem 0.3 to studying the fundamental group of the complement to the branch curve. It is easy to see that, if a $\mathcal{D}_{2n}$ covering $\pi : S \to \Sigma$ with $B$ as the branch locus exists, then there exists a surjective homomorphism from $\pi_1(\Sigma \setminus B) \to \mathcal{D}_{2n}$. This implies that $\pi_1(\Sigma \setminus B)$ is non-abelian. In particular, as we will see in §4, Assumption 0.2 holds when $\Sigma = \mathbf{P}^2$. Hence we have the following corollaries.

**Corollary 0.5.** *Let $p$ be a fixed odd prime. Let $B$ be a plane curve of degree $2m$ with at most simple singularities. Let $l$ be the integer as in Theorem 0.4. If $l > 4m^2 - 6m + 3 - \mu$, then there exists a $\mathcal{D}_{2p}$ covering, $S$, of $\mathbf{P}^2$ branched along $B$ with ramification index* 2. *In particular, $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.*

In particular, if $B$ has only nodes and cusps, we have the following:

**Corollary 0.6.** *Suppose that $B$ has $a$ nodes and $b$ cusps. If $2a + 6b > 2(2m)^2 - 6(2m) + 6$, then there exists a $\mathcal{D}_6$ covering, $S$, of $\mathbf{P}^2$ branched along $B$ with ramification index* 2. *In particular, $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.*

*Remark* 0.7. (i) The inequality in Corollary 0.5 is sharp in the following sense:

There exist pairs of irreducible plane sextic curves $(B_1, B_2)$ such that

(a) both $B_1$ and $B_2$ have the same combination of singularities; and it is either $3e_6$ or $e_6$, $4a_2$, $2a_1$, and

(b) there exists a $D_6$ covering branched along $B_1$ with ramification index 2, while there is no such covering for $B_2$.

For details, see [A], [T3] and [T5].

(ii) By [No], it is known that, if $2a + 6b < (2m)^2$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian. This contrast makes our result more interesting.

(iii) In [O], Oka recently showed that there exist sextic curves, $B_1$ and $B_2$, with 3 nodes and 6 cusps such that $\pi_1(\mathbf{P}^2 \setminus B_1)$ is non-abelian, while $\pi_1(\mathbf{P}^2 \setminus B_2)$ is abelian. This shows that the inequality Corollary 0.6 is the best possible.

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**Notations and Conventions.** In this paper, the ground field is always the complex number field $\mathbf{C}$.

Let $X$ be a normal variety, and let $Y$ be a smooth variety. Let $\pi : X \to Y$ be a finite morphism from $X$ to $Y$. We define the branch locus of $f$, which we denote by $\Delta(X/Y)$, as follows:

$$\Delta(X/Y) = \{y \in Y | \sharp(\pi^{-1}(y)) < \deg \pi\}.$$

Let $S$ be a finite double covering of a smooth projective surface $\Sigma$. The "*canonical resolution*" of $S$ always means the resolution given by Horikawa in [H].

For simple singularities of curves, we use the notations, $a$, $d$ and $e$, to describe the types of such singularities. Note that the capital letters, $A$, $D$ and $E$ are used for the corresponding types in [BPV].

Let $D_1, D_2$ be divisors.
$D_1 \sim D_2$: linear equivalence of divisors.
$D_1 \sim_{\mathbf{Q}} D_2$: $\mathbf{Q}$-linear equivalence of divisors.
$D_1 \approx D_2$: algebraic equivalence of divisors.
$D_1 \approx_{\mathbf{Q}} D_2$: $\mathbf{Q}$-algebraic equivalence of divisors.

## 1. Preliminaries

This section consists of 3 subsections. All are summaries on the known facts and results, which we need to prove Theorems 0.3 and 0.4.

**1. Summary on $\mathcal{D}_{2n}$ ($n$: odd) coverings.** In this article, we only consider the case that the Galois group is $\mathcal{D}_{2n}$, ($n$: odd).

We denote by $\mathcal{D}_{2n}$ a dihedral group of order $2n$, which is generated by two elements $\sigma$ and $\tau$ with the relations $\sigma^2 = \tau^n = (\sigma\tau)^2 = 1$.

Let $\pi : X \to Y$ be a $\mathcal{D}_{2n}$ covering. Let $\mathbf{C}(X)^\tau$ be the invariant subfield of $\mathbf{C}(X)$ by $\tau$. $\mathbf{C}(X)^\tau$ is a quadratic extension of $\mathbf{C}(Y)$. Let $D(X/Y)$ be the $\mathbf{C}(X)^\tau$-normalization of $Y$. $D(X/Y)$ is a double covering of $Y$ satisfying a commutative diagram as follows:

$$\begin{array}{ccc} X & & \\ & \searrow \beta_2 & \\ \downarrow \pi & & D(X/Y) \\ & \swarrow \beta_1 & \\ Y & & \end{array}$$

where $\beta_1 : D(X/Y) \to Y$ is a double covering of $Y$ and $\beta_2 : X \to D(X/Y)$ is an $n$-fold cyclic covering of $D(X/Y)$. Note that $\Delta(D(X/Y)/Y) = \Delta(X/Y)$ if $\pi$ is a $\mathcal{D}_{2n}$ covering branched along $B$ with ramification index 2. We fix these notations throughout this article. The following proposition is fundamental in constructing a $\mathcal{D}_{2n}$ covering.

**Proposition 1.1.** *Let $n$ be an odd integer $\geq 3$. Let $f : Z \to Y$ be a smooth finite double covering of $Y$. We denote the covering transformation by $\sigma$. Let $D$ be an effective divisor such that*

*(i) $D$ and $\sigma^* D$ have no common component,*

*(ii) if we let $D = \sum_i a_i D_i$ be the irreducible decomposition, then for all $i$, $a_i > 0$ and the greatest common divisor of $a_i$'s and $n$ is 1, and*

*(iii) there exists a line bundle $L$ such that $D - \sigma^* D \approx nL$.*

*Then there exists a $\mathcal{D}_{2n}$ covering, $X$, of $Y$ such that*

*(a) $D(X/Y) = Z$ and (b) the branch locus of $\beta_2$, $\Delta(X/Z)$, is contained in $\mathrm{Supp}(D + \sigma^* D)$, i.e., $\Delta(X/Y) \subset \Delta(Z/Y) \cup f(\mathrm{Supp}(D))$.*

*Proof.* It is enough to find the three effective divisors $D_1$, $D_2$ and $D_3$ on $Z$ satisfying the three conditions in Proposition 0.4 in [T1]. We first define $D_1$ in the following way:

For the coefficient of each irreducible component $D_i$ of $D$, put $a_i' = a_i - n[\frac{a_i}{n}]$, where $[\ ]$ denotes the greatest integer function. As $n \nmid a_i$, $a_i' > 0$. We rewrite

$D - \sigma^* D$ as follows:

$$\begin{aligned} D - \sigma^* D &= \sum_i \Big( a'_i D_i - a'_i \sigma^* D_i + n[\frac{a_i}{n}](D_i - \sigma^* D_i) \Big) \\ &= \sum_{i, a'_i \leq \frac{n-1}{2}} \Big( a'_i D_i - a'_i \sigma^* D_i + n[\frac{a_i}{n}](D_i - \sigma^* D_i) \Big) \\ &+ \sum_{i, a'_i > \frac{n-1}{2}} \Big( (n - a'_i)\sigma^* D_i - (n - a'_i) D_i + n([\frac{a_i}{n}] + 1)(D_i - \sigma^* D_i) \Big) . \end{aligned}$$

Put

$$D_1 = \sum_{i, a'_i \leq \frac{n-1}{2}} a'_i D_i + \sum_{i, a'_i > \frac{n-1}{2}} (n - a'_i)\sigma^* D_i.$$

(Note that the greatest common divisor of the coefficients, $a'_i$'s, and $n$ is 1 by the assumption (ii).) Since $\mathrm{Pic}^0(Z)$ is $n$-divisible, by replacing $L$ by a suitable one, we may assume that $D - \sigma^* D \sim nL$. Hence we have

$$D_1 - \sigma^* D_1 \sim n \left( L - \sum_{i,\, a'_i \leq \frac{n-1}{2}} [\frac{a_i}{n}](D_i - \sigma^* D_i) - \sum_{i,\, a'_i > \frac{n-1}{2}} ([\frac{a_i}{n}] + 1)(D_i - \sigma^* D_i) \right) .$$

Since any line bundle is linear equivalent to a difference of two effective divisors, the right hand side in the above is linear equivalent to a divisor of form $n(D_3 - D_2)$ where $D_2$ and $D_3$ are effective divisors. Thus we have three effective divisors $D_1$, $D_2$ and $D_3$ satisfying the three conditions in Proposition 0.4.

**2. Preliminary from lattice theory.** Here we summarize some elementary facts from lattice theory.

Let $L$ be a lattice, i.e.,

(i) $L$ is a free finite $\mathbf{Z}$ module and

(ii) $L$ is equipped with a non-degenerate bilinear symmetric pairing $\langle\ ,\ \rangle$.

For a given lattice $L$, $discL$ is the determinant of the intersection matrix. Note that it is independent of choice of a basis. We call $L$ unimodular if $discL = \pm 1$.

Let $J$ be a sublattice of $L$. We denote its orthogonal complement with respect to $\langle\ ,\ \rangle$ by $J^\perp$.

For a lattice $L$, we denote its dual lattice by $L^\vee$. Note that, by using the pairing, $L$ is embedded in $L^\vee$ as a sublattice with same rank. Hence the quotient group $L^\vee/L$ is a finite abelian group, which we denote by $G_L$.

Although the following lemma is often used under the assumption that $L$ is *even* unimodular, the statement also holds for any unimodular lattice (see [E]).

**Lemma 1.2.** *Let $L$ be a unimodular lattice . Let $J_1$ and $J_2$ be sublattices of $L$ such that $J_1^\perp = J_2$ and $J_2^\perp = J_1$. Then*

$$G_{J_1} \cong G_{J_2}.$$

A sublattice $M$ of $L$ is called primitive if $L/M$ is torsion-free.

**Example 1.3.** Let $S$ be a surface. Since $H^2(S, \mathbf{Z})/\,\mathrm{NS}(S)$ is torsion free, $\mathrm{NS}(S)$ is torsion free if and only if $H^2(S, \mathbf{Z})$ is. Therefore for the surface $Z$ in Theorems 0.3 and 0.4, $H^2(Z, \mathbf{Z})$ is torsion free. In fact, it is a unimodular lattice with respect to the intersection pairing, and $\mathrm{NS}(S)$ is a primitive sublattice of it.

**3. Analysis of the torsion part of** $\mathrm{NS}(Z)/T$**.** We use the same notations as those in Introduction. Throughout this subsection, we always assume that $\mathrm{NS}(Z)$ is torsion free. We start with the following lemma.

**Lemma 1.4.** $(\mathrm{NS}(Z)/T)_{\mathrm{tor}} = T^{\perp\perp}/T$.

*Proof.* By definition, $(\mathrm{NS}(Z)/T)_{\mathrm{tor}} \subset T^{\perp\perp}/T$, and we have $T^{\perp\perp} \subset \mathrm{NS}(Z)$ by Example 1.3. This implies our lemma.

$T^{\perp\perp}$ is considered as a sublattice of $T^{\vee}$ by using the intersection pairing. By Lemma 1.4, this means that the torsion subgroup of $\mathrm{NS}(Z)/T$ is considered as a subgroup of $G_T$. We next look at $G_T$.

Let $x$ be a singular point of $B$. Put

$R_x$ = the subgroup of $T$ generated by all irreducible components of the exceptional divisor of $f^{-1}(x)$.

Then we have a decomposition of $T$ as follows:

$$T \cong \tilde{f}^* \mathrm{NS}(\Sigma) \oplus \bigoplus_{x \in \mathrm{Sing}(B)} R_x.$$

With this decomposition, we have $T^{\vee}/T \cong G_{\tilde{f}^* \mathrm{NS}(\Sigma)} \oplus \bigoplus_{x \in \mathrm{Sing}(B)} G_{R_x}$. Note that, for each $G_{R_x}$, we have the following classical fact:

**Fact 1.5.** Since we only consider a torsion element of odd order, the cases of $a_n$ and $e_6$ are all we need. We consider both $R_x$ and $R_x^{\vee}$ as subgroups of $R_x \otimes \mathbf{Q}$ and give a $\mathbf{Q}$-divisor that gives rise to a generator of $G_{R_x}$. To this purpose, we label irreducible components of the exceptional divisors for $A_n$ and $E_6$ singularities (see Figure 1).

| The type of $x$ | The type of ${f'}^{-1}(x)$ | $G_{R_x}$ |
|---|---|---|
| $a_n$ | $A_n$ | $\mathbf{Z}/(n+1)\mathbf{Z}$ |
| $d_n, n \equiv 1 \bmod 2$ | $D_n$ | $\mathbf{Z}/4\mathbf{Z}$ |
| $d_n, n \equiv 0 \bmod 2$ | $D_n$ | $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ |
| $e_6$ | $E_6$ | $\mathbf{Z}/3\mathbf{Z}$ |
| $e_7$ | $E_7$ | $\mathbf{Z}/2\mathbf{Z}$ |
| $e_8$ | $E_8$ | $\{0\}$ |

**Lemma 1.6.** *$G_{R_x}$ is generated by the class of $\mathbf{Q}$-divisors $D_x/(b+1)$ (resp. $\frac{D_x}{3}$) for $x = a_b$ (resp. $x = e_6$), where*

$$D_x = \begin{cases} \sum_{k=1}^{\frac{b}{2}} (b+1-k)(\Theta_k - \Theta_{b-k}), & \text{if } x = a_b \ b\text{: even,} \\ \sum_{k=1}^{\frac{b-1}{2}} (b+1-k)(\Theta_k - \Theta_{b-k}) + \frac{b+1}{2}\Theta_{\frac{b+1}{2}} & \text{if } x = a_b \ b\text{: odd,} \end{cases}$$

*and*

$$D_x = (\Theta_1 - \Theta_5) + 2(\Theta_2 - \Theta_6) \quad \text{if } x = e_6.$$

*Proof.* The inverse of the intersection matrix of $R_x$ shows that a generator is given by the class of $\mathbf{Q}$-divisors

$$\frac{1}{b+1}\left(b\Theta_1 + (b-1)\Theta_2 + \cdots + \Theta_b\right) \quad \text{for } x = a_b,$$

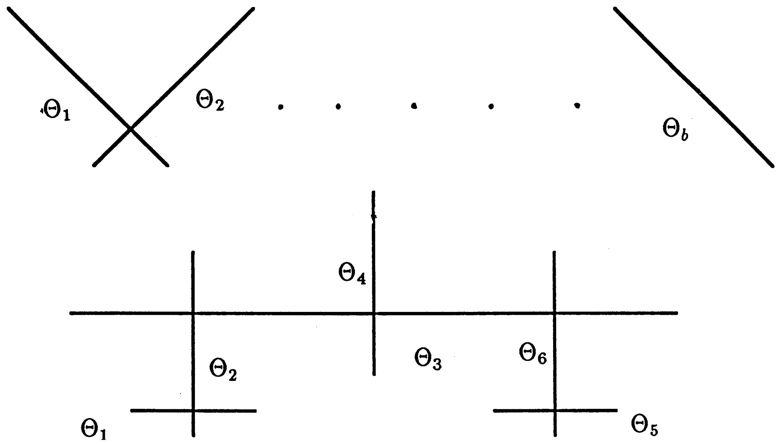


FIGURE 1.

and

$$\frac{1}{3}(4\Theta_1 + 5\Theta_2 + 6\Theta_3 + 3\Theta_4 + 2\Theta_5 + 4\Theta_6) \quad \text{for } x = e_6.$$

Our statement is straightforward from these $\mathbf{Q}$-divisors.

*Remark* 1.7. Let $\sigma$ denote the covering transformation of the double covering $f : Z \to \hat{\Sigma}$. Then we have $\sigma^*\Theta_k = \Theta_{n-k}$ $(1 \leq k \leq [\frac{n}{2}])$ for the exceptional divisor of an $A_n$ singularity and $\sigma^*\Theta_1 = \Theta_5$ and $\sigma^*\Theta_2 = \Theta_6$ for those of an $E_6$ singularity by the construction of the canonical resolution.

## 2. PROOF OF THEOREM 0.3

We first reduce the existence of a dihedral covering of $\Sigma$ to that of $\hat{\Sigma}$. Suppose that there exists a $\mathcal{D}_{2n}$ covering, $\hat{S}$, of $\hat{\Sigma}$ such that
(i) $D(\hat{S}/\hat{\Sigma}) = Z$, and
(ii) $\Delta(\hat{S}/Z) \subset$ *the support of the exceptional divisors of* $g : Z \to Z'$.

Then the Stein factorization, $S$, of $q \circ \hat{\pi} : \hat{S} \to \Sigma$ gives rise to the desired $\mathcal{D}_{2n}$ covering. Hence by Proposition 1.1 we only need to prove the following:

**Proposition 2.1.** *There exist a divisor $D$ and a line bundle $L$ such that*
*(i)* $\mathrm{Supp}(D + \sigma^*D) \subset \mathrm{Supp}($*the exceptional divisor of* $g)$, *and*
*(ii) the pair $(D, L)$ satisfies the condition in Proposition 1.1.*

We need several steps to prove Proposition 2.1.

Let $\nu$ be the homomorphism $T^{\perp\perp} \to T^{\vee} \to G_T$. Let $L_1$ be an element of $T^{\perp\perp}$ which gives rise to an $n$-torsion in $T^{\perp\perp}/T$. As $G_T \cong G_{\tilde{f}^*\,\mathrm{NS}(\Sigma)} \oplus \bigoplus_{x \in \mathrm{Sing}(B)} G_{R_x}$, we write

$$\nu(L_1) = \big(\alpha, (\beta_x)_{x \in \mathrm{Sing}(B)}\big) \in G_{\tilde{f}^*\,\mathrm{NS}(\Sigma)} \oplus \bigoplus_{x \in \mathrm{Sing}(B)} G_{R_x}.$$

**Lemma 2.2.** $\alpha = 0$.

*Proof.* As $\sharp G_{\tilde{f}^*\,\mathrm{NS}(\Sigma)} = \mathrm{disc}\,\tilde{f}^*\,\mathrm{NS}(\Sigma)$, $\alpha = 0$ by the assumption.

**Lemma 2.3.** *If $x$ is neither $a_n$ nor $e_6$, $\beta_x = 0$.*

*Proof.* This is immediate by Fact 1.5.

For simplicity, we put $r_x = \sharp(G_{R_x})$ in the following. Note that $r_x = b+1$ (resp. 3) if $x$ is of type $a_b$ (resp. $e_6$).

**Lemma 2.4.** *Suppose $\beta_x \neq 0$. Let $s_x$ be the order of $\beta_x$. As $s_x$ is a common divisor of $n$ and $s_x$, we put $r_x = s_x t_x$ and $n = s_x u_x$. Then there exists an integer $k_x$, $0 < k_x < s_x$, $(k_x, s_x) = 1$ such that*

$$\beta_x = \text{ the class of } \frac{k_x}{s_x} D_x,$$

*where $D_x$ is the divisor in Lemma 1.6.*

*Proof.* As $\beta_x \neq 0$, $x$ is of either type $a_n$ or $e_6$. In such a case, $G_{R_x}$ is a cyclic group with a generator as in Lemma 1.6, from which our lemma follows.

By Lemma 2.4, we have

$$L_1 \approx_{\mathbf{Q}} \sum_{x \in \mathrm{Sing}(B)} \frac{k_x}{s_x} D_x \bmod T.$$

This means that there exists an element, $L_2$, in $T$ such that

$$L_1 + L_2 \approx_{\mathbf{Q}} \sum_{x \in \mathrm{Sing}(B)} \frac{k_x}{s_x} D_x.$$

**Lemma 2.5.** *The greatest common divisor of the $(nk_x/s_x)$'s is* 1.

*Proof.* Let $d = \gcd\left((\frac{nk_x}{s_x})_{x \in \mathrm{Sing}\, B}\right)$. Then $\frac{n}{d}(L_1 + L_2) \in T$ as $\mathrm{NS}(Z)$ is torsion free. This shows that the order of $\nu(L_1 + L_2) = \nu(L_1)$ is a divisor of $\frac{n}{d}$. By our assumption, the order of $\nu(L_1)$ is $n$; and $d = 1$.

Now we define the divisor $D$ on $Z$ as follows:
If $\beta_x \neq 0$, $x = a_b$, $b$: even, then put

$$D_x^+ = \frac{nk_x}{s_x} \sum_{k=1}^{\frac{b}{2}} (b+1-k)\Theta_k.$$

If $\beta_x \neq 0$, $x = a_b$, $b$: odd, then put

$$D_x^+ = \frac{nk_x}{s_x} \sum_{k=1}^{\frac{b-1}{2}} (b+1-k)\Theta_k.$$

If $\beta_x \neq 0$, $x = e_6$, then put

$$D_x^+ = \frac{nk_x}{3}\Theta_1 + \frac{2nk_x}{3}\Theta_2.$$

Now put $D = \sum_{x \in \mathrm{Sing}(B)} D_x^+$. Then

$$D - \sigma^* D \approx n(L_1 + L_2 - \sum_{x = a_b, b:odd, \beta_x \neq 0} \frac{k_x r_x}{2 s_x} \Theta_{\frac{r_x}{2}}).$$

Put $L = L_1 + L_2 - \sum_{x = a_b, b:odd, \beta_x \neq 0} \frac{k_x r_x}{2s_x} \Theta_{\frac{r_x}{2}}$. Then the pair $(D, L)$ satisfies the condition in Proposition 2.1.

## 3. Proof of Theorem 0.4.

Thanks to Theorem 0.3, it is enough to show that $\mathrm{NS}(Z)/T$ has a $p$-torsion. We first prove the following lemma:

**Lemma 3.1.** *Let $b_i(Z)$ be the $i$-th Betti number of $Z$. Then we have*

$$b_2(Z) = 24\chi(\mathcal{O}_\Sigma) + 4(h^1(\mathcal{O}_\Sigma) + h^1(\mathcal{O}_\Sigma(-\mathcal{L}))) + 4\mathcal{L}^2 + 2K_\Sigma\mathcal{L} - 2K_\Sigma^2 - 2.$$

*Proof.* By [H], Lemma 6 and its proof, we have

$$\chi(\mathcal{O}_Z) = \frac{1}{2}\mathcal{L}(K_\Sigma + \mathcal{L}) + 2\chi(\mathcal{O}_\Sigma),$$
$$K_Z^2 = 2(K_\Sigma + \mathcal{L})^2 \quad \text{and,}$$
$$h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_Z) + h^1(\mathcal{O}_Z(-\mathcal{L})).$$

Hence the Noether formula gives

$$c_2(Z) = 12\chi(\mathcal{O}_Z) - c_1^2(Z) = 24\chi(\mathcal{O}_Z) + 4\mathcal{L}^2 + 2K_\Sigma\mathcal{L} - 2K_\Sigma^2.$$

As $b_1(Z) = b_3(Z) = 2h^1(\mathcal{O}_Z)$, we have the desired equality.

In the following, we make use of some Nikulin theory ([N]). This approach is a modification of Miranda-Persson's in §4, [MP]. A similar argument is also found in [X].

Suppose that there exists no $p$-torsion in $T^{\perp\perp}/T$. Then

$$S_p(G_T) \cong S_p(G_{T^{\perp\perp}}),$$

where $S_p(G)$ denotes the $p$-Sylow group of $G$. By Lemma 1.6 and Example 1.7, we have

$$G_{T^\perp} \cong G_{T^{\perp\perp}}.$$

Hence the number of generators, $l_1$, of $S_p(G_{T^{\perp\perp}}) \le \operatorname{rank} T^\perp = b_2(Z) - \operatorname{rank} T$. On the other hand, by the assumption and Lemma 3.1, we have $l_1 \ge l > b_2(Z) - \operatorname{rank} T$. This leads us to a contradiction.

## 4. Applications

We keep the same notations as before.

*Application* 1: *$\mathcal{D}_{2n}$ coverings of $\mathbf{P}^2$.*

**Lemma 4.1.** *Suppose that*

(*i*) *$\Sigma$ is simply connected, and*

(*ii*) *the linear system $|B|$ is a base point and fixed component free.*

*Then $\pi_1(Z) = 0$. In particular,* $\mathrm{NS}(Z)$ *is torsion free.*

*Proof.* Let $B_1$ be a smooth member of $|B|$. Then there exists a smooth double covering $Z_1$ of $\Sigma$ branched along $B_1$. By Brieskorn's results for the simultaneous resolution of rational double points ([B1], [B2]), $Z$ is considered as a smooth deformation of $Z_1$. This implies that $Z$ is homeomorphic to $Z_1$. By Proposition 1.8, [C], $Z_1$ is simply connected. Hence $Z$ is simply connected.

In the case where $\Sigma = \mathbf{P}^2$, the two assumptions in Lemma 4.1 always hold. This gives Corollary 0.5

**Example 4.2.** (i) Put $p = 3$ and $m = 2$. Consider a plane quartic curve having $3a_2$ singularities. Then there exists a $\mathcal{D}_6$ covering branched along $B$ with the ramification index 2. This gives another proof for Zariski's result: $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian ([Z]).

(ii) Put $p = 3$ and $m = 3$. Let $B$ be the sextic curve as in the table in §6, [T4]. Then there exists a $\mathcal{D}_6$ covering branched along $B$ with the ramification index 2. Note that we proved the existence for only the first 7 cases in [T4].

*Remark* 4.3. The author does not know any single example of a plane curve of degree $\geq 8$ enjoying the assumption in Corollary 0.5. Also, he does not know of any single example of a plane curve enjoying the assumption in Corollary 0.5 for $p \geq 5$.

*Application* 2: *$n$-torsions of an elliptic surface and $\mathcal{D}_{2n}$ coverings*

In [T1], [T2] and [T4], $p$-torsions of the Mordell-Weil group of an elliptic surface play important roles in constructing $\mathcal{D}_{2p}$ ($p$: odd prime) coverings. We show that the results on the existence of $\mathcal{D}_{2p}$ coverings in previous articles follow easily from Theorem 0.4.

Let $\varphi : \mathcal{E} \to C$ be an elliptic surface over a curve $C$ such that

(i) $\varphi$ is relatively minimal,

(ii) $\varphi$ has a section $s_0$, and

(iii) $\varphi$ has at least one singular fiber.

Under these assumptions, by Theorem 1.2 in [S], $\mathrm{NS}(\mathcal{E})$ is torsion free. Also it is well-known that $\mathcal{E}$ is obtained as the canonical resolution of a double covering $f' : \mathcal{E}' \to \Sigma$ of some ruled surface $\Sigma$ over $C$ having the branch locus $\Delta(\mathcal{E}'/\Sigma)$ in the form of $\Delta_0 + B_0$ such that

(i) $\Delta_0$ is a section of $\Sigma$ and $B_0$ is a tri-section with $B_0 \cap \Delta_0 = \emptyset$,

(ii) the image of $s_0$ in $\Sigma$ is $\Delta_0$, and

(iii) $B_0$ has at most simple singularities.

We denote the morphism from $\mathcal{E} \to \Sigma$ by $\tilde{f}$.

We apply Theorem 0.3 to $\Sigma$ and $\Delta(\mathcal{E}'/\Sigma)$ and obtain the following theorem, which is again a generalization of Proposition 3.1 in [T4].

**Theorem 4.4.** *Let $n$ be an odd number. Let $MW(\mathcal{E})$ be the Mordell-Weil group of $\mathcal{E}$, i.e., the group of sections. If $MW(\mathcal{E})$ has an $n$-torsion, then there exists a $\mathcal{D}_{2n}$ covering, $S$, of $\Sigma$ branched along $\Delta(\mathcal{E}'/\Sigma)$ with ramification index 2.*

*Proof.* As $\operatorname{disc} \tilde{f}^* \mathrm{NS}(\Sigma) = 4$, $\gcd(n, \operatorname{disc} \tilde{f}^* \mathrm{NS}(\Sigma)) = 1$. Let $T$ be as before and let $T_\varphi$ be the subgroup of $\mathrm{NS}(\mathcal{E})$ generated by $s_0$, $F$: a fiber of $\varphi$, and all the irreducible components of singular fibers not meeting $s_0$. By our construction, the difference between $T$ and $T_\varphi$ is just one between $\tilde{f}^* \mathrm{NS}(\Sigma)$ and $\mathbf{Z}s_0 \oplus \mathbf{Z}F$. As $\tilde{f}^*\Delta_0 = 2s_0$ and a fiber of $\Sigma \to C$ is also a fiber of $\varphi$, $\tilde{f}^* \mathrm{NS}(\Sigma)$ is a subgroup of $\mathbf{Z}s_0 \oplus \mathbf{Z}F$ of index 2. This means that $[T_\varphi : T] = 2$; and we have the exact sequence

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathrm{NS}(\mathcal{E})/T \to \mathrm{NS}(\mathcal{E})/T_\varphi \to 0.$$

This implies that, for $n$: odd, $\mathrm{NS}(\mathcal{E})/T$ has an $n$-torsion if and only if $\mathrm{NS}(\mathcal{E})/T_\varphi$ does also. On the other hand, by Theorem 1.3 in [S], $\mathrm{NS}(\mathcal{E})/T_\varphi \cong MW(\mathcal{E})$. By Theorem 0.3, we have our statement. □

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